THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2015–2016)
Introduction to Topology
Exercise 6 Complete and Baire Category

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Do the exercises mentioned in lectures or in lecture notes.

2. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence such that the set $\{ x_n : n \in \mathbb{N} \}$ has a cluster point. What can you conclude about the sequence.

3. If both X and Y are complete metric spaces, is the product metric space $X \times Y$ complete? Note that there are many ways to define the product metric.

4. Let $\mathcal{B}[a, b]$ be the set of bounded functions on the interval $[a, b]$ and

$$d_\infty(f, g) = \sup_{t\in[a,b]} \|f(t) - g(t)\| \, .$$

   Show that $(\mathcal{B}[a, b], d_\infty)$ is a complete metric space.

5. Let $\mathcal{C}[a, b]$ be the set of continuous functions on the interval $[a, b]$ and

$$d_1(f, g) = \int_a^b |f(t) - g(t)| \, dt \, .$$

   Show that $(\mathcal{C}[a, b], d_1)$ is not complete.

6. Explore the possible relation between a contraction mapping and a one-to-one mapping.

7. Let $f\colon X \to Y$ be uniformly continous and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Show that $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

8. Let d be a metric on a space X and $x_0 \in X$. Is the function $f(x) = d(x, x_0)$ uniformly continous?

9. In a discrete space, find all the dense sets and all the nowhere dense sets.

10. Show that the followings are equivalent:
    - A is dense
    - The only open set contained in $X \setminus A$ is $\emptyset$
    - The only closed set containing A is X

11. Let $N \subset X$ be nowhere dense. Show that every open set $U \subset X$ contains an open set $V \subset U$ such that $V \cap N = \emptyset$.

12. Show that $\mathbb{Z}$ with the standard metric $d(m, n) = |m - n|$ is of second category. Note: this does not contradict that $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

13. Show that if $\{ N_k \}_{k=1}^n$ is a finite family of nowhere dense sets, then $\bigcup_{k=1}^n N_k$ is also nowhere dense.

14. Let X be of second category. If $\{ N_k \}_{k\in\mathbb{N}}$ is a countable family of nowhere dense sets, then there exists a point $x \in X$ such that $x \notin \bigcup_{k\in\mathbb{N}} N_k$.

15. Are there statements about first and second category of $X \times Y$ with reference to the categories of X and Y?

16. Show that $A \subset X$ is open dense if and only if $X \setminus A$ is closed nowhere dense. Give counter examples if the open/closed condition is dropped.

17. Let $f\colon\ X \to Y$ be a continuous mapping.

    (a) If $D \subset X$ is dense, is $f(D) \subset Y$ dense?

    (b) If $N \subset X$ is nowhere dense, is $f(N) \subset Y$ nowhere dense?

    (c) What about pre-images of a dense set and a nowhere dense set?

    (d) What can you conclude about image or pre-image of a set of first or second category?